RC 14492 (64907) 3/22/89, revised 7/15/96
Mathematics 9 pages

Research Report

Some theoretical results concerning L-moments

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Abstract. L-moments and L-moment ratios are quantities useful in the summarization and estimation of probability distributions. Hosking (*J. R. Statist. Soc. B*, 1990) describes the theory and applications of L-moments. Here we give an expanded discussion of some of the theory in Hosking (1990), including in particular proofs of Theorems 2.1, 2.2 and 2.3 of that paper.

1. L-moments: definitions and basic properties

Let X be a real-valued random variable with cumulative distribution function $F(x)$ and quantile function $x(F)$, and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the order statistics of a random sample of size n drawn from the distribution of X. Define the *L-moments* of X to be the quantities

$$\lambda_r \equiv r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} EX_{r-k:r}\,, \qquad r = 1, 2, \ldots\,. \tag{1}$$

The L in "L-moments" emphasizes that λ_r is a *linear* function of the expected order statistics. Furthermore, as noted in Hosking (1990, section 3), the natural estimator of λ_r based on an observed sample of data is a linear combination of the ordered data values, i.e. an L-statistic. The expectation of an order statistic may be written as

$$EX_{j:r} = \frac{r!}{(j-1)!\,(r-j)!} \int x\{F(x)\}^{j-1}\{1-F(x)\}^{r-j}\,dF(x)$$

(David, 1981, p. 33). Substituting this expression in (1), expanding the binomials in $F(x)$ and summing the coefficients of each power of $F(x)$ gives

$$\lambda_r = \int_0^1 x(F)\,P^*_{r-1}(F)\,dF\,, \qquad r = 1, 2, \ldots, \tag{2}$$

where

$$P^*_r(F) = \sum_{k=0}^{r} p^*_{r,k}\,F^k$$

and

$$p^*_{r,k} = (-1)^{r-k} \binom{r}{k} \binom{r+k}{k}\,.$$

$P^*_r(F)$ is the rth shifted Legendre polynomial, related to the usual Legendre polynomials $P_r(u)$ by $P^*_r(u) = P_r(2u-1)$. Shifted Legendre polynomials are orthogonal on the interval $(0,1)$ with constant weight function (Lanczos, 1957, p. 286—though his $P^*_r(\,.\,)$ differs by a factor $(-1)^r$ from ours). The first few L-moments are

$$\begin{aligned}
\lambda_1 &= EX &&= \int x\,.\,dF\,,\\
\lambda_2 &= \tfrac{1}{2}E(X_{2:2}-X_{1:2}) &&= \int x\,.\,(2F-1)\,dF\,,\\
\lambda_3 &= \tfrac{1}{3}E(X_{3:3}-2X_{2:3}+X_{1:3}) &&= \int x\,.\,(6F^2-6F+1)\,dF\,,\\
\lambda_4 &= \tfrac{1}{4}E(X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4}) &&= \int x\,.\,(20F^3-30F^2+12F-1)\,dF\,.
\end{aligned}$$

The use of L-moments to describe probability distributions is justified by the following theorem.

Theorem 1. (i) The L-moments λ_r, $r=1,2,\ldots$, of a real-valued random variable X exist if and only if X has finite mean.

(ii) A distribution whose mean exists is characterized by its L-moments $\{\lambda_r : r = 1,2,\ldots\}$.

Proof. A finite mean implies finite expectations of all order statistics (David, 1981, p. 33), whence part (i) follows immediately.

For part (ii), we first show that a distribution is characterized by the set $\{EX_{r:r}, r=1,2,\ldots\}$. This has been proved by Chan (1967) and Konheim (1971): the following proof is essentially Konheim's. Let X and Y be random variables with cumulative distribution functions F and G and quantile functions $x(u)$ and $y(u)$ respectively. Let

$$\xi_r^{(X)} \equiv EX_{r:r} = r \int x\{F(x)\}^{r-1} dF(x)\,, \qquad \xi_r^{(Y)} \equiv EY_{r:r} = r \int x\{G(x)\}^{r-1} dG(x)\,.$$

Then

$$\begin{aligned} \xi_{r+2}^{(X)} - \xi_{r+1}^{(X)} &= \int_0^1 \{(r+2)u^{r+1} - (r+1)u^r\}\, x(u)\, du \\ &= \int_0^1 u^r \,.\, u(1-u)\, dx(u) \qquad \text{by parts} \\ &= \int_0^1 u^r \,.\, dz_X(u) \end{aligned}$$

where $z_X(u)$, defined by $dz_X(u) = u(1-u)dx(u)$, is an increasing function on $(0,1)$. If $\xi_r^{(X)} = \xi_r^{(Y)}$, $r=1,2,\ldots$, then

$$\int_0^1 u^r\, dz_X(u) = \int_0^1 u^r\, dz_Y(u)\,, \qquad r = 0,1,\ldots\,.$$

Thus z_X and z_Y are distributions which have the same moments on the finite interval (0,1); consequently (Feller, 1970, pp. 222–224), $z_X = z_Y$. This implies that $x(u) = y(u)$.

We have shown that a distribution with finite mean is characterized by the set $\{\xi_r : r = 1,2,\ldots\}$. Using (2), we have

$$\lambda_r = \sum_{k=1}^{r} p^*_{r-1,k-1} k^{-1} \xi_k\,,$$

whence

$$\xi_r = \sum_{k=1}^{r} \frac{(2k-1)\, r!\, (r-1)!}{(r-k)!\, (r-1+k)!} \lambda_k\,. \qquad (3)$$